OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MATHEMATICS 2640

Mechanics 4

Monday **21 JANUARY 2002** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
List of Formulae (MF8)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- Where a numerical value for the acceleration due to gravity is needed, use $9.8\,\text{m}\,\text{s}^{-2}$.
- You are permitted to use a graphic calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 60.
- Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
- **You are reminded of the need for clear presentation in your answers.**

This question paper consists of 3 printed pages and 1 blank page.

1 A wheel rotating about a fixed axis is slowing down with constant angular deceleration. Initially the angular speed is 24 rad s^{-1}. In the first 5 seconds the wheel turns through 96 radians.

(i) Find the angular deceleration. [2]

(ii) Find the total angle the wheel turns through before coming to rest. [2]

2 A uniform solid of revolution is formed by rotating the region bounded by the x-axis, the line $x = 1$ and the curve $y = x^2$ for $0 \leqslant x \leqslant 1$, about the x-axis. The units are metres, and the density of the solid is 5400 kg m^{-3}. Find the moment of inertia of this solid about the x-axis. [5]

3 A uniform rectangular lamina $ABCD$ of mass 0.6 kg has sides $AB = 0.4$ m and $AD = 0.3$ m. The lamina is free to rotate about a fixed horizontal axis which passes through A and is perpendicular to the lamina.

(i) Find the moment of inertia of the lamina about the axis. [3]

(ii) Find the approximate period of small oscillations in a vertical plane. [3]

4 A uniform circular disc has mass m, radius a and centre C. The disc is free to rotate in a vertical plane about a fixed horizontal axis passing through a point A on the disc, where $CA = \frac{1}{3}a$.

(i) Find the moment of inertia of the disc about this axis. [2]

The disc is released from rest with CA horizontal.

(ii) Find the initial angular acceleration of the disc. [2]

(iii) State the direction of the force acting on the disc at A immediately after release, and find its magnitude. [4]

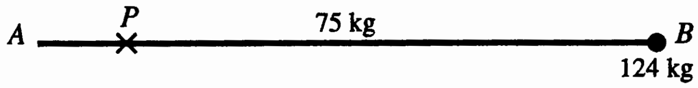
5 The region bounded by the x-axis, the y-axis, the line $x = \ln 5$ and the curve $y = e^x$ for $0 \leqslant x \leqslant \ln 5$, is occupied by a uniform lamina.

(i) Show that the centre of mass of this lamina has x-coordinate

$$\frac{5}{4}\ln 5 - 1.$$ [5]

(ii) Find the y-coordinate of the centre of mass. [3]

6



An arm on a fairground ride is modelled as a uniform rod AB, of mass 75 kg and length 7.2 m, with a particle of mass 124 kg attached at B. The arm can rotate about a fixed horizontal axis perpendicular to the rod and passing through the point P on the rod, where $AP = 1.2$ m.

(i) Show that the moment of inertia of the arm about the axis is $5220\,\text{kg m}^2$. [3]

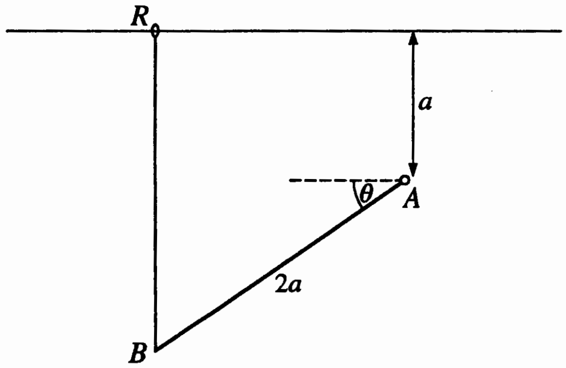
(ii) The arm is released from rest with AB horizontal, and a frictional couple of constant moment 850 N m opposes the motion. Find the angular speed of the arm when B is first vertically below P. [5]

7 At midnight, ship A is 70 km due north of ship B. Ship A travels with constant velocity $20\,\text{km h}^{-1}$ in the direction with bearing 140°. Ship B travels with constant velocity $15\,\text{km h}^{-1}$ in the direction with bearing 025°.

(i) Find the magnitude and direction of the velocity of A relative to B. [4]

(ii) Find the distance between the ships when they are at their closest, and find the time when this occurs. [5]

8



The diagram shows a uniform rod AB, of mass m and length $2a$, free to rotate in a vertical plane about a fixed horizontal axis through A. A light elastic string has natural length a and modulus of elasticity $\frac{1}{2}mg$. The string joins B to a light ring R which slides along a smooth horizontal wire fixed at a height a above A and in the same vertical plane as AB. The string BR remains vertical. The angle between AB and the horizontal is denoted by θ, where $0 < \theta < \pi$.

(i) Taking the reference level for gravitational potential energy to be the horizontal through A, show that the total potential energy of the system is

$$mga(\sin^2\theta - \sin\theta).$$ [3]

(ii) Find the three values of θ for which the system is in equilibrium. [5]

(iii) For each position of equilibrium, determine whether it is stable or unstable. [4]

1

$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ (equivalent of the linear $s = ut + \frac{1}{2})at^2$)

$96 = 5 \times 24 + \frac{1}{2}\alpha \times 25$

$\alpha = {}^{-}$ rad

so the angular decelleration is $\mathbf{1 \cdot 92}$ **rad s^{-2}** (show)

[2]

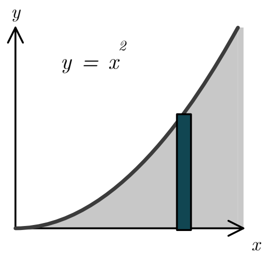
$\omega_1{}^2 = \omega_0{}^2 + 2\alpha\theta$ ("$v^2 = u^2 + 2as$")

$0 = 24^2 + 2 \times {}^{-}1 \cdot 92 \times \theta$

$\boldsymbol{\theta} = \mathbf{150}$ **rad**

[2]

2



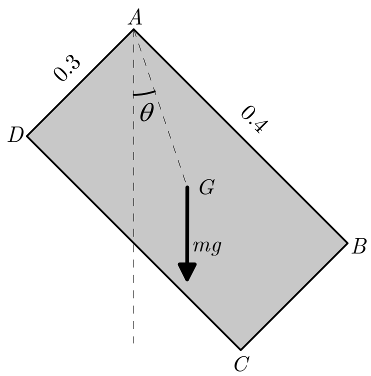
mass of 'elemental disc' $= \rho . \pi y^2 . \delta x = 5400\pi x^4 \delta x$

MoI of 'elemental disc' $= \frac{1}{2}mr^2 = 2700\pi x^8 \delta x$

$I = \int_0^1 \left(2700\pi x^8\right) \mathrm{d}x = 300\pi = \mathbf{942}$ **kg m^2** (3 s.f.)

[5]

3



$I_{AB} = \frac{4}{3}ml^2 = \frac{4}{3} \times 0 \cdot 6 \times 0 \cdot 15^2 = 0 \cdot 018$

$I_{AD} = \frac{4}{3}ml^2 = \frac{4}{3} \times 0 \cdot 6 \times 0 \cdot 20^2 = 0 \cdot 032$

perpendicular axes rule ....

$I_A = 0 \cdot 018 + 0 \cdot 032 = \mathbf{0 \cdot 05}$ **kg m^2**

[3]

$AG = 0 \cdot 25$ (Pythagoras)

when AG makes angle θ with the vertical ...

$$I\ddot{\theta} = {}^{-}mg(0 \cdot 25 \sin\theta)$$

$$\ddot{\theta} = {}^{-}29 \cdot 4 \sin\theta$$

and so for small θ ...

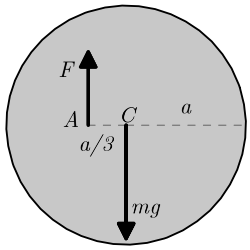
$$\ddot{\theta} \approx {}^{-}29 \cdot 4\theta$$

so the motion is approximately SHM with

$$T = \frac{2\pi}{\sqrt{29 \cdot 4}} = 1 \cdot 15879... = \mathbf{1 \cdot 16\ s} \quad \text{(3 s.f.)}$$

[3]

4



parallel axes rule …

$$I_A = I_C + m\left(\tfrac{1}{3}a\right)^2 = \tfrac{1}{2}ma^2 + \tfrac{1}{9}ma^2 = \boldsymbol{\tfrac{11}{18}ma^2}$$

[2]

on release …

$$\text{M}(A) \qquad C = I\ddot{\theta}$$

$$mg\left(\tfrac{1}{3}a\right) = \left(\tfrac{11}{18}ma^2\right)\ddot{\theta}$$

$$\boldsymbol{\ddot{\theta} = \tfrac{6g}{11a}}$$

[2]

When released from rest no 'central' force is needed to maintain circular motion about A so **the force on the disc a A is purely vertical (upwards)**.

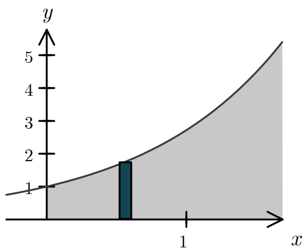
[1]

$$\text{N2}(\downarrow) \qquad mg - F = m\left(r\ddot{\theta}\right)$$

$$F = mg - m\left(\tfrac{1}{3}a\right)\left(\tfrac{6g}{11a}\right) = \boldsymbol{\tfrac{9}{11}mg}$$

[3]

5



$$m = \rho\int_0^{\ln 5} e^x \,\mathrm{d}x = 4\rho$$

$$4\rho\,\overline{x} = \int_0^{\ln 5} \rho\, xe^x \,\mathrm{d}x = \rho\left[xe^x\right]_0^{\ln 5} - \rho\int_0^{\ln 5} e^x \mathrm{d}x = \rho.5\ln 5 - 4\rho$$

$$\boldsymbol{\overline{x} = \tfrac{5}{4}\ln 5 - 4} \quad \text{(show)}$$

[5]

using the same 'strips' …

$$4\rho\overline{y} = \int_0^{\ln 5} \left(\tfrac{1}{2}y\right).\rho y \,\mathrm{d}x = \int_0^{\ln 5} \tfrac{1}{2}\rho e^{2x} \mathrm{d}x = \rho\left[\tfrac{1}{4}e^{2x}\right]_0^{\ln 5} = \tfrac{1}{4}\rho(25-1) = 6\rho$$

$$\boldsymbol{\overline{y} = 1\cdot 5}$$

[3]

6

$$I = 124\times 6^2 + \left(\tfrac{1}{3}\times 75\times 3\cdot 6^2 \;+\; 75\times 2\cdot 4^2\right) = 4464 + 756 = \mathbf{5220\ \ kg\,m^2} \qquad \text{(show)}$$

[3]

energy considerations …

gain in K.E. = loss in G.P.E. − work done by frictional couple

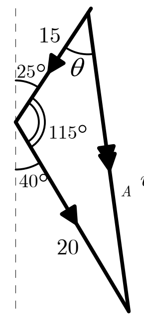
$$\tfrac{1}{2}I\omega^2 = mgh - C\theta$$

$$2610\omega^2 = 75\times 9\cdot 8\times 2\cdot 4 - 850\times\tfrac{\pi}{2}$$

$$\boldsymbol{\omega = 1\cdot 72}\ \ \mathbf{rad\,s^{-1}}$$

[5]

7 ${}_A\mathbf{v}_B = {}_A\mathbf{v}_G - {}_B\mathbf{v}_G$



$$\left|{}_A\mathbf{v}_B\right|^2 = 20^2 + 15^2 - 2\times 20\times 15\times\cos 115^\circ = 878\cdot 640...$$

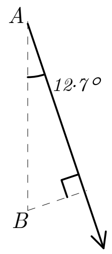
$$\left|{}_A\mathbf{v}_B\right| = 29\cdot 6406... = \mathbf{29\cdot 6\ \ km\,h^{-1}} \qquad \text{(3 s.f.)}$$

$$\frac{\sin\theta}{20} = \frac{\sin 115^\circ}{29\cdot 640...}$$

$$\theta = 37\cdot 7^\circ$$

the relative velocity is on **bearing 167°** [4]

now considering the **position** of A relative to B .....



closest distance $= 70\sin 12\cdot 7^\circ = \mathbf{15\cdot 4\ \ km}$

$$t = \frac{70\cos 12\cdot 7^\circ}{29\cdot 64} = 2\cdot 30\,\text{hrs}$$

ships closest together at 2:18 am

[5]

8 relative to $\theta = 0$ position ...

$$V = {}^{-}mg\left(a\sin\theta\right) + \tfrac{1}{2}\times\frac{\frac{1}{2}mg}{a}\left(2a\sin\theta\right)^2 = \boldsymbol{mga}\left(\mathbf{sin^2}\boldsymbol{\theta} - \mathbf{sin}\,\boldsymbol{\theta}\right) \qquad \text{(show)}$$

[3]

$$\frac{\mathrm{d}V}{\mathrm{d}\theta} = mga\left(2\sin\theta\cos\theta - \cos\theta\right) = mga\cos\theta\left(2\sin\theta - 1\right)$$

so for $0 < \theta < \pi$ the system is in equilibrium for $\boldsymbol{\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}}$ [5]

$$\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2} = mga\left(\sin\theta - 2\sin^2\theta + 2\cos^2\theta\right)$$

$\theta = \frac{\pi}{6}$ $\qquad \frac{\mathrm{d}^2V}{\mathrm{d}\theta^2} = \frac{3}{2}mga > 0$ $\qquad \therefore$ **stable**

$\theta = \frac{\pi}{2}$ $\qquad \frac{\mathrm{d}^2V}{\mathrm{d}\theta^2} = {}^{-}mga < 0$ $\qquad \therefore$ **unstable**

$\theta = \frac{5\pi}{6}$ $\qquad \frac{\mathrm{d}^2V}{\mathrm{d}\theta^2} = \frac{3}{2}mga > 0$ $\qquad \therefore$ **stable**

[4]